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VIBRATION OF INHOMOGENEOUS STRINGS, RODS AND MEMBRANES

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This paper is concerned with the vibration of inhomogeneous strings, rods and membranes with continuously varying properties. The work has two main purposes. First we provide specific examples for which closed-form exact solutions are obtained. Apart from their intrinsic interest, such examples serve as benchmark problems against which the accuracy of approximate analytical or numerical methods may be assessed. The second objective is to demonstrate the effectiveness of an integral-equation-based method for obtaining lower bounds for vibration frequencies.

1. INTRODUCTION

Vibration problems for inhomogeneous structural elements present a challenge to the acoustics and vibration community. In contrast to classical problems for homogeneous strings, rods, membranes or plates for which exact solutions and a variety of approximate methods are available for finding frequencies and mode shapes, analogous results for inhomogeneous bodies are relatively scarce. A number of recent papers in this journal [1–5] have been concerned with free transverse vibrations of non-homogeneous membranes with continuously varying density or thickness variations. Several approximate techniques for such problems are considered in references [1–4] while reference [5] provides some exact solutions. Earlier work for *composite* membranes, with discontinuous property changes, has been described in references [3, 6, 7] and references cited therein.

The purpose of the present paper is two-fold. First we provide further specific examples of inhomogeneous vibrating strings, rods or membranes for which closed-form exact solutions are obtained. As pointed out in reference [5], apart from their intrinsic interest, such examples serve as benchmark problems against which the accuracy of approximate analytic or numerical methods may be assessed. Our second objective is to demonstrate the effectiveness of an integral equation approach for obtaining lower bounds for vibration frequencies. In contrast to upper bounds, for which Rayleigh–Ritz methods and their variants have been well documented, accurate lower bounds for vibration frequencies are much more

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difficult to obtain. The integral-equation-based method has already been shown (see, e.g., references [6, 8, 9]) to be more effective than classical differential equation techniques for obtaining such bounds for vibration problems with discontinuous coefficients. Here we demonstrate a similar result for problems with continuously varying properties.

The plan of the paper is as follows. In the next section, we describe three vibration problems for inhomogeneous strings and rods whose mathematical description is equivalent. All three problems can be cast in the form of a special Sturm–Liouville eigenvalue problem (see equations (7) and (8)). Such problems have been investigated recently [10, 11] in an entirely different context, namely, the study of end effects for anti-plane shear deformations of laterally inhomogeneous isotropic elastic materials. The main results of the paper are presented in Section 3. First we provide four specific examples of inhomogeneities for which exact closed-form solutions are obtained. The fundamental frequencies are given and their dependence on the degree of inhomogeneity is examined. *In particular, it is shown that certain non-homogeneous strings and rods have the same fundamental frequencies as their homogeneous counterparts.* Secondly, the integral-equation-based method for obtaining lower bounds for the fundamental frequency is described and its accuracy is demonstrated for the four illustrative examples. Extension of the results to non-homogeneous membranes is outlined in Section 4.

2. INHOMOGENEOUS STRINGS AND RODS

Here we describe three vibration problems for inhomogeneous strings and rods whose mathematical descriptions are equivalent. Our treatment follows that of Gladwell [12].

2.1. PROBLEM 1

The free transverse vibration of a stretched string of length l, subject to constant tension τ, with mass per unit length $m(x)$, deflection $u(x)$, and vibrating with frequency ω is described by

$$u''(x) + \lambda m(x)u(x) = 0 \quad \text{on } 0 < x < l, \tag{1}$$

where $\lambda = \omega^2/\tau$ and $' = \mathrm{d}/\mathrm{d}x$. The simplest boundary conditions are those of a fixed end, in which case

$$u(0) = u(l) = 0. \tag{2}$$

2.2. PROBLEM II

The longitudinal vibration (in which each cross-section moves only in the x direction) of a thin straight rod of length l, cross-sectional area $A(x)$, constant density ρ, deflection $w(x)$, and constant Young's modulus E is governed by

$$[A(x)w'(x)]' + \lambda A(x)w(x) = 0 \quad \text{on } 0 < x < l, \tag{3}$$

where $\lambda = \rho\omega^2/E$. The simplest boundary conditions occur when at $x = 0$ and $x = l$ there are fixed supports, so that

$$w(0) = w(l) = 0. \tag{4}$$

2.3. PROBLEM III

The free torsional vibrations of a thin straight rod of length l, with moment of inertia $J(x)$, twist $\theta(x)$, constant density ρ, and constant shear modulus G are governed by the equation

$$[J(x)\theta'(x)]' + \lambda J(x)\theta(x) = 0 \quad \text{on } 0 < x < l, \tag{5}$$

where $\lambda = \rho\omega^2/G$. The simplest end conditions are

$$\theta(0) = \theta(l) = 0. \tag{6}$$

It can be seen that the mathematical descriptions of Problems II and III are identical and are of the form

$$[\mu(x)g'(x)]' + \lambda\mu(x)g(x) = 0, \quad g(0) = g(l) = 0. \tag{7, 8}$$

It will be assumed henceforth that $\mu(x) > 0$ is continuously differentiable on $(0, l)$. The problem (7, 8) is a regular Sturm–Liouville eigenvalue problem of special form since the variable coefficient $\mu(x)$ is the same in both terms in equation (7). Thus, the extensive body of knowledge available for problem (7) and (8) may be directly applied to Problems II and III.

We now convert problem (7) and (8) to the Liouville normal form in order to show that Problem I is also equivalent to equations (7) and (8). In equations (7) and (8) set

$$\begin{aligned} T &= \int_0^l \mu^{-1}(s)\,\mathrm{d}s, \quad v(t) = g(x(t)), \\ t(x) &= T^{-1}\int_0^x \mu^{-1}(s)\,\mathrm{d}s, \quad f(t) = [T\mu(x(t))]^2. \end{aligned} \tag{9}$$

Then the λ are the eigenvalues of

$$v''(t) + \lambda f(t)v(t) = 0 \quad \text{on } 0 < t < l, \tag{10}$$

subject to

$$v(0) = v(1) = 0. \tag{11}$$

Here a prime denotes differentiation with respect to t. The problem (10) and (11) is again of Sturm–Liouville form, and is seen to be identical to Problem I.

3. FUNDAMENTAL FREQUENCIES AND MODE SHAPES

3.1. EXACT RESULTS

In what follows, we provide some examples for the inhomogeneity $\mu(x)$ for which *exact* solutions of equations (7) and (8) can be found. Direct interpretations of these results for Problem II are immediate if $\mu(x)$ is identified as

$$\mu(x) = A(x) \tag{12}$$

and

$$g(x) = w(x), \quad \omega = (E\lambda/\rho)^{1/2}, \tag{13}$$

where the physical description of the quantities in equations (12) and (13) have been given in Section 2. Similarly for Problem III, one identifies $\mu(x)$ as

$$\mu(x) = J(x), \tag{14}$$

and

$$g(x) = \theta(x), \quad \omega = (G\lambda/\rho)^{1/2}. \tag{15}$$

The implications for Problem I require use of equation (9) and will be discussed later.

We consider *four* illustrative examples where the $\mu(x)$ in the differential equation (7) is taken to be that of Examples 1, 2, 3, 4 respectively in Table 1, where $\alpha \geqslant 0$, $\mu_0 > 0$ are arbitrary constants. The behaviour of $\mu(x)$ in Examples 1 and 4 is qualitatively similar in that for fixed α, the quantity $\mu(x)$ is monotonically decreasing in x and for fixed x, the quantity μ is monotonically decreasing in α in both examples. Likewise, Examples 2 and 3 are similar in that the corresponding quantities are monotonically increasing. When $\alpha = 0$, all the vibrating elements are homogeneous so that $\mu = \mu_0$ and the smallest eigenvalue λ_1 of problem (7) and (8) is

$$\lambda_1 = \pi^2/l^2, \tag{16}$$

TABLE 1

The example inhomogeneities in equation (7)

Example 1:	$\mu(x) = \mu_0\left(1 + \frac{\alpha x}{l}\right)^{-1}$
Example 2:	$\mu(x) = \mu_0\left(1 + \frac{\alpha x}{l}\right)$
Example 3:	$\mu(x) = \mu_0\left(1 + \frac{\alpha x}{l}\right)^2$
Example 4:	$\mu(x) = \mu_0 \exp(-\alpha x/l)$

with corresponding fundamental frequencies given by equations (13) and (15). The parameter α provides a measure of the "degree of inhomogeneity" of the string or rod.

For Example 1, it has been shown in reference [10] that the general solution of equation (7) can be written in terms of Bessel functions as

$$g(x) = \left(1 + \frac{\alpha x}{l}\right)\left\{A J_1\left[\frac{\lambda^{1/2} l}{\alpha}\left(1 + \frac{\alpha x}{l}\right)\right] + B Y_1\left[\frac{\lambda^{1/2} l}{\alpha}\left(1 + \frac{\alpha x}{l}\right)\right]\right\} \tag{17}$$

so that $\lambda_1^{1/2} l = \alpha s_1$, where $s_1 = s_1(\alpha)$ is the smallest positive root of

$$J_1(s) Y_1[(1 + \alpha)s] - Y_1(s) J_1[(1 + \alpha)s] = 0. \tag{18}$$

Similarly, for Example 2, one finds that

$$g(x) = A J_0\left[\frac{\lambda^{1/2} l}{\alpha}\left(1 + \frac{\alpha x}{l}\right)\right] + B Y_0\left[\frac{\lambda^{1/2} l}{\alpha}\left(1 + \frac{\alpha x}{l}\right)\right] \tag{19}$$

and so $\lambda_1^{1/2} l = \alpha s_1$, where $s_1 = s_1(\alpha)$ is the smallest positive root of

$$J_0(s) Y_0[(1 + \alpha)s] - Y_0(s) J_0[(1 + \alpha)s] = 0. \tag{20}$$

The smallest positive roots of the transcendental equations (18) and (20) may be found in standard tables. Plots of $\lambda_1^{1/2} l$ versus α are given in Figures 1 and 2. The corresponding mode shapes are given by equations (17) and (19), respectively, with λ replaced by λ_1. For Example 3, it has been shown in reference [10] that

$$\lambda_1^{1/2} l = \pi, \quad g_1(x) = B(1 + \alpha x/l)^{-1} \sin(\pi x/l). \tag{21}$$

This result shows that an inhomogeneous rod, with $A(x)$ given by Example 3 for longitudinal vibration or with $J(x)$ given by Example 3 for torsional vibrations, has the same fundamental frequency as the corresponding uniform rod with constant properties.

For Example 4, it can be shown (see reference [11]) that

$$\lambda_1^{1/2} l = (4\pi^2 + \alpha^2)^{1/2}/2, \quad g_1(x) = B \exp(\alpha x/2l) \sin(\pi x/l). \tag{22}$$

A plot of $\lambda_1^{1/2} l$ versus α is given in Figure 3. We see that the fundamental frequency increases monotonically with α. As is discussed in reference [10, 11], the foregoing is true for the class of inhomogeneities for which $(\mu^{1/2})'' \geqslant 0$ and

$$C(x, \alpha) \equiv [\mu^{1/2}(x, \alpha)]'' \mu^{-1/2}(x, \alpha) \tag{23}$$

is monotonically increasing in α. Expression (23) provides a measure of the convexity of $\mu(x, \alpha)$ as a function of x. It may be easily verified that $C(x, \alpha)$ is indeed monotonically increasing in α for Examples 1 and 4. Similarly, it can be shown that the fundamental frequency decreases monotonically with α if $(\mu^{1/2})'' \leqslant 0$ and $-C(x, \alpha)$ is monotonically increasing in α. These conditions can be shown to hold for Example 2. For Example 3, $(\mu^{1/2})'' \equiv 0$.

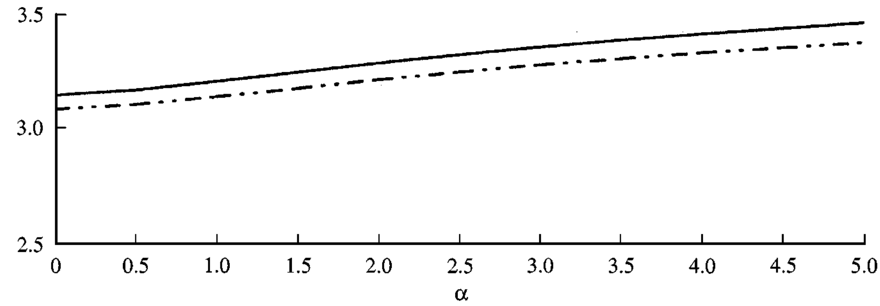

Figure 1. Exact value of $\lambda_1^{1/2} l$ (——) and its lower bound (—··—) for Example 1.

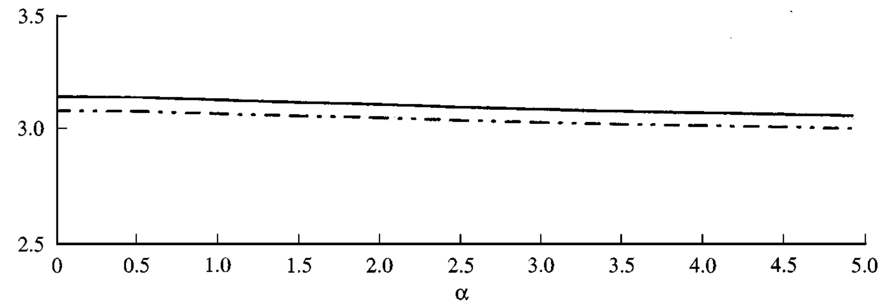

Figure 2. Exact value of $\lambda_1^{1/2} l$ (——) and its lower bound (—··—) for Example 2.

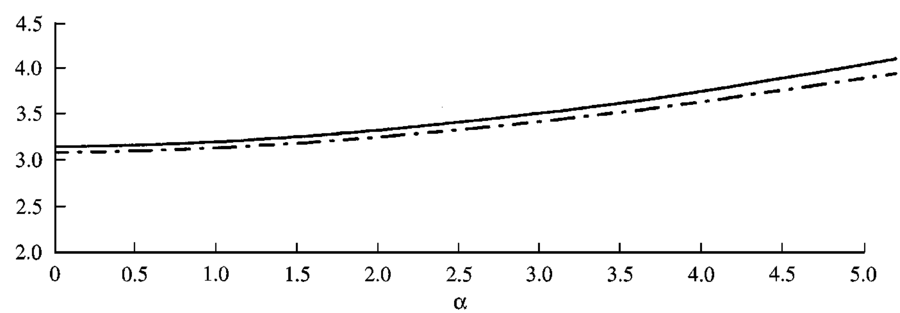

Figure 3. Exact value of $\lambda_1^{1/2} l$ (——) and its lower bound (—··—) for Example 4.

We turn now to application of the preceding exact solutions of problem (7) and (8) to Problem I for the vibrating string. By using equation (9), the function $f(t)$ appearing in equation (10) may be calculated. By comparing equation (10) with equation (1), one finds that the mass per unit length $m(x)$ of the string is given as in Table 2. Thus, a vibrating string with mass per unit length given as in Table 2 and subjected to constant tension τ, has the fundamental frequency

$$\omega_1 = (\tau\lambda_1)^{1/2}, \tag{24}$$

TABLE 2

The function m(x) in equation (1)

Example 1:	$m(x) = \dfrac{(\alpha+2)^2}{4}\left[(\alpha^2+2\alpha)\dfrac{x}{l}+1\right]^{-1}$
Example 2:	$m(x) = \left[\dfrac{\ln(1+\alpha)}{\alpha}\right]^2 (1+\alpha)^{2x/l}$
Example 3:	$m(x) = (\alpha+1)^2\left(\alpha+1-\dfrac{\alpha x}{l}\right)^{-4}$
Example 4:	$m(x) = \dfrac{(e^\alpha-1)^2}{\alpha^2}\left[1+(e^\alpha-1)\dfrac{x}{l}\right]^{-2}$

where the exact values of $\lambda_1^{1/2}$ are given by equations (18), (20), (21) and (22), respectively. *In particular, the inhomogeneous string with mass per unit length given by Example 3 of Table 2 has the same fundamental frequency as the corresponding uniform string with constant properties.*

3.2. BOUNDS ON VIBRATION FREQUENCIES

Since exact solutions of the problem (7) and (8) are rarely available, numerous techniques have been developed in the literature for approximating the eigenvalues and corresponding eigenfunctions. For *upper* bounds on λ, the Rayleigh–Ritz method and its many variants are the most widely used. Accurate *lower* bounds on λ are much more difficult to obtain. In what follows, we briefly describe a technique based on an integral equation approach for obtaining lower bounds on λ_1. This method was developed in references [6, 8, 9] for problems of the form (7) and (8) where the $\mu(x)$ in equation (7) is discontinuous. Such problems arise in many areas in the mechanics of composites. The integral-equation-based method has been shown in references [6, 8, 9] to be particularly suitable for such problems. As we shall demonstrate using Examples 1–4, the integral-equation method also provides extremely accurate lower bounds for λ_1 when $\mu(x)$ is continuous as in the present paper.

We begin with the formulation (10) and (11). In reference [8] it is shown that equations (10) and (11) are equivalent to the Fredholm integral equation of the second kind,

$$V(t) = \lambda \int_0^1 \Delta(t,t_0)V(t_0)\,\mathrm{d}t_0, \quad V(t) = v(t)\sqrt{f(t)}, \tag{25}$$

where the symmetric kernel of the integral equation is defined as

$$\Delta(t,t_0) = f^{1/2}(t)G(t,t_0)f^{1/2}(t_0), \tag{26}$$

and the Green's function $G(t, t_0)$ is

$$G(t,t_0) = \begin{cases} (1-t_0)t & \text{on } 0 \leqslant t \leqslant t_0, \\ (1-t)t_0 & \text{on } t_0 \leqslant t \leqslant 1. \end{cases} \tag{27}$$

The L^2 norm of Δ is given by

$$\|\Delta\|^2 = \int_0^1 \int_0^{t_0} f(t) f(t_0)(1-t_0)^2 t^2 \,\mathrm{d}t\,\mathrm{d}t_0 + \int_0^1 \int_{t_0}^1 f(t) f(t)(1-t)^2 t_0^2 \,\mathrm{d}t\,\mathrm{d}t_0. \tag{28}$$

It can be shown, by a change of variables, that the double integrals in equation (28) are equal. Thus

$$\|\Delta\|^2 = 2\int_0^1 \int_0^{t_0} f(t) f(t_0)(1-t_0)^2 t^2 \,\mathrm{d}t\,\mathrm{d}t_0 \tag{29}$$

or, equivalently,

$$\|\Delta\|^2 = 2\int_0^1 \int_{t_0}^1 f(t) f(t)(1-t)^2 t_0^2 \,\mathrm{d}t\,\mathrm{d}t_0. \tag{30}$$

In reference [8] it is shown that a lower bound for the smallest positive eigenvalue of problem (10) and (11) is given by

$$\lambda_1 \geqslant \underline{\lambda_1} \equiv \|\Delta\|^{-1}. \tag{31}$$

The lower bound $\lambda_1 \geqslant \underline{\lambda_1}$ has been explicitly evaluated in references [10, 11] for Examples 1–4. The results are given in Table 3.

In Figures 1–3, these lower bounds for Examples 1, 2, and 4 are plotted, together with the exact values of $\lambda_1^{1/2} l$ given by equations (18), (20) and (22), respectively. As

TABLE 3

The lower bound $\underline{\lambda_1^{1/2}}$ from equation (31)

Example 1:	$\underline{\lambda_1^{1/2}}\, l = \dfrac{(8/\varphi)^{1/4}}{(2+\alpha)}$ where $\varphi = \dfrac{\ln(1+\beta)^2\,(1/2 + 1/\beta + 1/2\beta^2)}{\beta^4} + \dfrac{1/12 - 1/\beta - 1/\beta^2}{2\beta^2}$ and $\beta = \alpha(\alpha+2)$
Example 2:	$\underline{\lambda_1^{1/2}}\, l = \dfrac{\alpha\sqrt{8\ln(1+\alpha)}}{\chi^{1/4}}$ where $\chi = 2[(1+\alpha)^4 + 1]\{[\ln(1+\alpha)]^2 + 2\} - 5[(1+\alpha)^4 - 1]\ln(1+\alpha) - 8(1+\alpha)^2$
Example 3:	$\underline{\lambda_1^{1/2}}\, l = 3^{1/2}\, 10^{1/4} \cong 3{\cdot}08007$
Example 4:	$\underline{\lambda_1^{1/2}}\, l = \left(\dfrac{\alpha^4(\mathrm{e}^{\alpha}-1)^2}{2[-4(\mathrm{e}^{\alpha}-1)^2 + (\mathrm{e}^{2\alpha}-1)\alpha + 2\alpha^2\mathrm{e}^{\alpha}]}\right)^{1/4}$

$\alpha \to 0$, all of the lower bounds have the values $3^{1/2} \times 10^{1/4} = 3{\cdot}08007$, whereas the exact results have the value π. It is seen that the lower bounds are extremely accurate and reflect the monotonic character of the exact values as the parameter α varies. It is also striking that in the case of Example 3 (not plotted), for which the exact fundamental frequency (21) is π, the lower bound $\lambda_1^{1/2} l$ is also independent of α. *Thus, the lower bounds furnished by the integral-equation technique are seen to provide a remarkably accurate estimate for the fundamental frequency for the examples considered.* An additional advantage of the method is the fact that the results are explicit in the inhomogeneity measure α, thus allowing for parametric studies. The integral-equation-based method can also be used to find lower bounds for higher eigenfrequencies (see references [8, 9]) but we shall not pursue this here.

4. CONCLUDING REMARKS

We conclude with a brief outline of application of the preceding results to vibrations of inhomogeneous membranes. The transverse vibrations of a membrane occupying the simply connected plane domain D and fixed on the boundary ∂D are described by

$$\nabla^2 w + \lambda \rho(x, y) w = 0 \quad \text{on } D \tag{32}$$

subject to

$$w = 0 \quad \text{on } \partial D. \tag{33}$$

Here $w(x, y)$ is the transverse displacement, $\rho(x, y)$ the mass density and

$$\lambda = \omega^2/\tau, \tag{34}$$

where τ is the constant tension per unit length and ω the frequency of vibration.

Suppose that D is the rectangular domain $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$. If one seeks separable solutions of equation (32) of the form

$$w(x, y) = f(x) g(y), \tag{35}$$

then it is easily shown that, if the mass density $\rho(x, y)$ is such that

$$\rho(x, y) = \rho_1(x) + \rho_2(y), \tag{36}$$

one obtains a pair of Sturm–Liouville problems of the form

$$f''(x) + \gamma p(x) f(x) = 0, \tag{37}$$

$$f(0) = 0, \qquad f(a) = 0, \tag{38}$$

$$\ddot{g}(y) + \eta q(y) g(y) = 0, \tag{39}$$

$$g(0) = 0, \qquad g(b) = 0, \tag{40}$$

where

$$\lambda\rho(x, y) = \gamma p(x) + \eta q(y), \tag{41}$$

and where the overdot denotes differentiation with respect to y. Each of the preceding eigenvalue problems with eigenvalue parameter γ and η, respectively, has the form (1) and (2). Thus, the exact solutions developed in Section 3 for the densities of Table 2 may be used to generate exact solutions for inhomogeneous rectangular membranes with densities of the form (36), where ρ_1 and ρ_2 can be taken equal to *any* entry in Table 2. Similarly, the lower-bound techniques described in Section 3 can be used to obtain lower bounds for the membrane frequencies.

A special case of the foregoing has been considered in recent papers in this journal [1–5] where ρ_2 in equation (36) has been taken to be *constant*, and $\rho_1(x)$ is assumed to be *linear* in x. Approximate methods are employed in references [1–4] while an exact solution is obtained in reference [5].

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